MTH 201: Multivariable Calculus and Differential Equations

Homework IV

(Due 27/09)

1. Find the local maximum and minimum and saddle points of the following functions.
   (a) $f(x,y) = x^4 + y^4 - 4xy + 2$.
   (b) $f(x,y) = e^x \cos y$.
   (c) $f(x,y) = x^2 + y^2 + \frac{1}{x^2y^2}$.
2. Find the absolute maximum and minimum values values of f on the set D.
   (a) $f(x,y) = x^2 + y^2 + x^2y + 4$, $D = \{(x,y) \,|\, |x| \le 1, |y| \le 1\}$.
   (b) $f(x,y) = 2x^3 + y^4$, $D = \{(x,y) \,|\, x^2 + y^2 \le 1\}$.
   (c) $f(x,y) = xy^2$, $D = \{(x,y) \,|\, x \ge 0, y \ge 0, x^2 + y^2 \le 3\}$.
3. Find the shortest distance between the point P and the surface S.
   (a) $P = (1,2,3)$, $S = \{(x,y,z) \,|\, x - y + z = 4\}$.
   (b) $P = (0,0,0)$, $S = \{(x,y,z) \,|\, z^2 = xy + 1\}$.
4. Find the volume of the largest rectangular box with edges parallel to the axes that can be inscribed in the ellipsoid
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$
5. Find the dimensions of a rectangular box of largest volume such that the sum of the lengths of its 12 edges is the constant c.
6. Use Lagrange multipliers to find the maximum and minimum values of the given functions subject to the given constraints.
   (a) $f(x,y) = x^2 + y^2$; $x^4 + y^4 = 1$
   (b) $f(x,y,z) = x^2y^2z^2$; $x^2 + 2y^2 + 3z^2 = 6$.
   (c) $f(x,y,z) = 3x - y - 3z$; $x + y - z = 0$, $x^2 + 2z^2 = 1$.
   (d) $f(x_1, \ldots, x_n) = x_1 + \ldots + x_n$; ${x_1}^2 + \ldots + {x_n}^2 = 1$.
7. Use Lagrange multipliers to prove that the triangle with maximum area that has a given perimeter p is equilateral.
8. Find the highest and lowest points on the ellipse, which is the intersection of the plane $4x - 3y + 8z = 5$ and the cone $z^2 = x^2 + y^2$.
9. (a) Find the maximum value of
$$f(x_1, \ldots, x_n) = \sqrt[n]{x_1x_2\ldots x_n},$$
   where $x_1, x_2, \ldots, x_n$ are positive numbers and $x_1 + \ldots + x_n = c$.

   (b) Deduce form part (a) that if $x_1, \ldots, x_n$ are positive numbers, then
$$\sqrt[n]{x_1x_2\ldots x_n} \le \frac{x_1 + \ldots + x_n}{n}.$$

10. The two equations $x+y=uv$ and $xy=u-v$ determine x and v as functions of u and y say $x=X(u,y)$ and $v=V(u,y)$. Show that $\frac{\partial X}{\partial u}=\frac{u+v}{1+yu}$.

11. The equation $f(y/x,z/x)=0$ defines z implicitly as a function of x and y, say $z=g(x,y)$. Show that

$$x\frac{\partial g}{\partial x}+y\frac{\partial g}{\partial y}=g(x,y)$$

at those points at which $D_2f(y/x,g(x,y)/x)\neq 0$.

12. The equation $\sin(x+y)+\sin(y+z)=1$ determines z implicitly as a function of x and y, say $z=f(x,y)$. Compute $D_{1,2}f$ in terms of x, y, and z.

13. The three equations $F(u,v)=0$, $u=xy$, and $v=\sqrt{x^2+y^2}$ define a surface in $\mathbb{R}^3$. Find a normal vector to this surface at $(1,1,\sqrt{3})$ if it is known that $D_1F(1,2)=1$ and $D_2F(1,2)=2$.